On Some Representations of Nilpotent Lie Algebras and Superalgebras

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Abstract

Let G be a simply connected, nilpotent Lie group with Lie algebra $\mathfrak{g}$. The group G acts on the dual space $\mathfrak{g}^*$ by the coadjoint action. By the orbit method of Kirillov, the simple unitary representations of G are in bijective correspondence with the coadjoint orbits in $\mathfrak{g}^*$, which in turn are in bijective correspondence with the primitive ideals of the universal enveloping algebra of $\mathfrak{g}$. The number of simple $\mathfrak{g}$-modules which have a common eigenvector for a particular subalgebra of $\mathfrak{g}$ and are annihilated by a particular primitive ideal I is shown by Benoist to depend on geometric properties of a certain subvariety of the coadjoint orbit corresponding to I. We determine the exact number of such modules when the coadjoint orbit is two-dimensional.

Bell and Musson showed that the algebras obtained by factoring the universal enveloping superalgebra of a Lie superalgebra by graded-primitive ideals are isomorphic to tensor products of Weyl algebras and Clifford algebras. We describe certain cases where the factors are purely Weyl algebras and determine how the sizes of these Weyl algebras depend on the graded-primitive ideals.

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1 Introduction

The study of the representations of a real Lie group G is related to the study of the representations of its complex Lie algebra $\mathfrak{g}$. If G is simply connected

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and nilpotent, then the irreducible unitary representations of G are related to certain ideals of the universal enveloping algebra of $\mathfrak{g}$. This correspondence links ideas from geometry, analysis and algebra. The representation theory of Lie superalgebras is similar to, yet different from the representation theory of Lie algebras. Lie superalgebras are of interest to physicists in the context of supergauge symmetries relating particles of different statistics. This paper is devoted to the study of some aspects of the representation theory of nilpotent Lie algebras and superalgebras.

The orbit method was created by Kirillov in the attempt to describe the unitary dual $\hat{N}_m$ for the nilpotent Lie group N_m of $m \times m$ upper triangular matrices with 1's on the diagonal (the unitriangular group). It turned out that the orbit method had much wider applications. In Kirillov's words: '...all main questions of representation theory of Lie groups: construction of irreducible representations, restriction-induction functors, generalized and infinitesimal characters, Plancherel measure, etc., admit a transparent description in terms of coadjoint orbits' ([Kir03]).

For a nilpotent Lie algebra $\mathfrak{g}$, Dixmier in [Dix96] formulated the correspondence between the set of primitive ideals of the universal enveloping algebra $U(\mathfrak{g})$, the set of coadjoint orbits in $\mathfrak{g}^*$, and the sizes of the Weyl algebras obtained by factoring $U(\mathfrak{g})$ by primitive ideals. In general, for any primitive ideal I, there are infinitely many non-isomorphic simple $\mathfrak{g}$-modules that have I as their annihilator. Benoist in [Ben90b] used the orbit method to show, for a finite-dimensional nilpotent Lie algebra $\mathfrak{g}$, that the number (of isomorphism classes) of simple $\mathfrak{g}$-modules which are annihilated by a primitive ideal I *and* have a common eigenvector for a certain subalgebra of $\mathfrak{g}$, depends on properties of a certain subvariety of the coadjoint orbit corresponding to I under the Dixmier correspondence. We refine his results in a particular case.

The methods used by Dixmier and Conze ([Con71]) to describe the primitive ideals of universal enveloping algebras have been extended to study graded-primitive ideals of universal enveloping superalgebras of Lie superalgebras by Letzter ([Let92]), and by Bell and Musson ([BM90], [Mus92]). In [BM90] it is shown that the algebras obtained by factoring the universal enveloping superalgebra by graded-primitive ideals are isomorphic to tensor products of Weyl algebras and Clifford algebras. We describe certain cases where the factors are purely Weyl algebras and determine how the sizes of these Weyl algebras depend on the graded-primitive ideals.

2 Basic Definitions for Lie Algebras

Our objects of study in sections 2 through 9 are finite-dimensional nilpotent Lie algebras $\mathfrak{g}$ over the field $\mathbb{C}$ of complex numbers. Thus if

$$\mathfrak{g}^0 = \mathfrak{g}, \ \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \ \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1], \dots, \ \mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i].$$

is the lower central series of $\mathfrak{g}$, then $\mathfrak{g}^k = 0$ for some $k \geq 1$.

By the Birkhoff embedding theorem ([CG90, Thm. 1.1.11]), any nilpotent Lie algebra of finite dimension over $\mathbb{C}$ is isomorphic to a subalgebra of $\mathfrak{n}_m$ for some m, where $\mathfrak{n}_m$ is the Lie algebra of strictly upper triangular $m \times m$ matrices under the product $[x, y] = xy - yx$ for $x, y \in \mathfrak{n}_m$.

Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. If exp is the exponential map from $\mathfrak{g}$ to G, then $\exp(\mathfrak{g}) = G$ ([CG90, Thm. 1.2.1]). In fact, if we identify $\mathfrak{g}$ with a subalgebra of $\mathfrak{n}_m$, then the exponential map becomes the ordinary exponential map $x \mapsto \sum_{j=0}^{\infty} \frac{1}{j!} x^j$. The group G acts on $\mathfrak{g}$ by the *adjoint* action,

$$\mathrm{Ad}g(x) = gxg^{-1}, \ \ \forall x \in \mathfrak{g}, \ g \in G,$$

and it acts on the dual space $\mathfrak{g}^*$ by the *coadjoint* action. Thus, if $f \in \mathfrak{g}^*$, then

$$(g.f)(y) = f(g^{-1}yg), \ \ \ \forall g \in G, y \in \mathfrak{g}.$$

The orbit of $f \in \mathfrak{g}^*$ under the action of G is called the *coadjoint orbit* containing f. It is denoted by Ω_f.

3 Primitive Ideals and Weyl Algebras

An ideal I of the universal enveloping algebra $U = U(\mathfrak{g})$ is said to be *primitive* if it is the annihilator of a simple left $\mathfrak{g}$-module. The set of primitive ideals of $U(\mathfrak{g})$ is denoted by Prim U. For a nilpotent Lie algebra $\mathfrak{g}$, any primitive ideal of $U(\mathfrak{g})$ is maximal among the set of proper two-sided ideals of $U(\mathfrak{g})$ by [Dix96, Prop. 4.7.4].

Definition 3.1. *For $n \geq 1$, the n-th* Weyl algebra $\mathcal{A}_n$ *is the algebra with $2n$ generators $p_1, q_1, \dots, p_n, q_n$, and relations*

$$[p_i, q_i] = 1,$$

$$[p_i, q_j] = [p_i, p_j] = [q_i, q_j] = 0 \ \ for \ i \neq j.$$

By convention $\mathcal{A}_0 = \mathbb{C}$.

By [Dix96, Prop. 4.7.9], if I is a primitive ideal of $U(\mathfrak{g})$, then the quotient $U(\mathfrak{g})/I$ is isomorphic to $\mathcal{A}_n$, for some positive integer n. This integer n is uniquely determined by the ideal I, and it is called the *weight* of I ([Dix96, 4.7.10]).

4 Coadjoint Orbits and Primitive Ideals

Any $f \in \mathfrak{g}^*$ determines an alternating bilinear form B_f on $\mathfrak{g}$ given by

$$(x, y) \mapsto B_f(x, y) := f([x, y]) \quad \forall x, y \in \mathfrak{g}.$$

Let

$$\begin{aligned}
\mathfrak{g}^f &= \{x \in \mathfrak{g} \mid f([x, y]) = 0 \ \forall y \in \mathfrak{g}\} \\
&= \{x \in \mathfrak{g} \mid B_f(x, y) = 0 \ \forall y \in \mathfrak{g}\}.
\end{aligned}$$

It is obvious that $\mathfrak{g}^f$ is a subalgebra of $\mathfrak{g}$ by the Jacobi identity on $\mathfrak{g}$. It is called the *radical* of f, or the *kernel* of the form B_f.

A Lie subalgebra $\mathfrak{k}$ is said to be *subordinate* to f if $f([x, y]) = 0$ for all $x, y \in \mathfrak{k}$, i.e. if $\mathfrak{k}$ is a totally isotropic subspace of $\mathfrak{g}$ with respect to the alternating bilinear form B_f. The largest dimension of a subalgebra subordinate to f is $\frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}^f)$ by [Dix96, 1.12.1]. A subalgebra that is subordinate to f and has this maximal dimension is called a *polarisation* of f. The set of all polarisations of f is denoted by $P(f)$. If $\mathfrak{p} \in P(f)$, then $\mathfrak{p} \supseteq \mathfrak{g}^f$ (see [Dix96, 1.12.1]).

Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ and let W be an $\mathfrak{h}$-module. Since $U(\mathfrak{g})$ is a right $U(\mathfrak{h})$-module under multiplication, we can form the induced module $V = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} W$ with $U(\mathfrak{g})$-action given by

$$x(u \otimes w) = xu \otimes w \quad \forall x, u \in U(\mathfrak{g}) \text{ and } w \in W.$$

Assume $f \in \mathfrak{g}^*$ and $\mathfrak{p} \in P(f)$. Then, $f([x, y]) = 0$ for all $x, y \in \mathfrak{p}$, so we can define a $\mathfrak{p}$-module action on a one-dimensional vector space $\{\mathfrak{p}, f\} = \mathbb{C}v$ as follows:

$$x.v = f(x)v, \quad \forall x \in \mathfrak{p}.$$

Then $Ind_{\mathfrak{p}}^{\mathfrak{g}}\{\mathfrak{p}, f\} = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}v$, the $\mathfrak{g}$-module induced from $\{\mathfrak{p}, f\}$, is a simple $\mathfrak{g}$-module, hence a simple $U(\mathfrak{g})$-module by [Dix96, Thm. 6.1.1]. The annihilator in $U(\mathfrak{g})$ of this module is a primitive ideal, denoted by $I(f)$. By [Dix96, Thm. 6.1.4], the ideal $I(f)$ depends only on f, and not on the choice of the polarisation $\mathfrak{p} \in P(f)$. Any primitive ideal I is of the form $I(f)$, for some $f \in \mathfrak{g}^*$ ([Dix96, Sec. 6.1.5, Thm. 6.1.7]).

If $f \in \mathfrak{g}^*$ and $a \in G$, then $I(a.f) = I(f)$ by [Dix96, Prop. 2.4.17]. Thus, the map

$$f \mapsto I(f),$$

between $\mathfrak{g}^*$ and Prim U, defines a map

$$\Omega_f \mapsto I(f)$$

between the set of G-coadjoint orbits in $\mathfrak{g}^*$ and Prim U. By [Dix96, Thm. 6.1.7, Prop. 6.2.3, Thm. 6.2.4], this map is a bijection. Moreover, we have $U(\mathfrak{g})/I(f) \simeq \mathcal{A}_n$, where $n = \frac{1}{2}\mathrm{rank}(B_f) = \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{g}^f)$ according to [Dix96, Prop. 6.2.2]. Notice that if $\mathfrak{g}$ is a finite-dimensional nilpotent Lie algebra, and if $f \in \mathfrak{g}^*$, then the weight of the associated primitive ideal $I(f)$ equals $\dim(\mathfrak{g}/\mathfrak{p})$ for any polarisation $\mathfrak{p}$ of f.

5 Geometry of Coadjoint Orbits

First we recall some generalities about symplectic vector spaces and symplectic varieties, and then we describe the symplectic structure on a coadjoint orbit.

A symplectic structure on an even-dimensional vector space V is determined by a non-degenerate alternating bilinear form ω on V. If W is a subspace of V, then its *orthogonal complement* is

$$W^\perp = \{v \in V \mid \omega(w, v) = 0 \ \forall w \in W\}.$$

If $W \subseteq W^\perp$, then W is called *isotropic*. If $W^\perp \subseteq W$, then W is called *coisotropic*. A subspace W that is both isotropic and coisotropic is said to be a *lagrangian* subspace. A lagrangian subspace of V is always of dimension $\frac{1}{2}\dim V$ (see [Cou95, Ch. 11, Prop. 2.2]).

Let $\mathcal{V}$ be an algebraic variety. A symplectic structure on $\mathcal{V}$ is a non-degenerate algebraic 2-form ω on $\mathcal{V}$ such that $d\omega = 0$ (see [CG97, Sec. 1.1] for details). If $p \in \mathcal{V}$, then there is a alternating bilinear form ω_p on $T_p\mathcal{V}$, the tangent space at p. If $\mathcal{W}$ is a subvariety of $\mathcal{V}$, then it is said to be *lagrangian* if the tangent space $T_p\mathcal{W}$ is a lagrangian subspace of $T_p\mathcal{V}$ at every non-singular point $p \in \mathcal{V}$. The dimension $\dim \mathcal{V}$ of a variety $\mathcal{V}$ is defined to be the dimension of the tangent space $T_p\mathcal{V}$ at any non-singular point $p \in \mathcal{V}$. Thus, if $\mathcal{W}$ is a lagrangian subvariety of $\mathcal{V}$, then $\dim \mathcal{W} = \frac{1}{2}\dim \mathcal{V}$.

Any coadjoint orbit $\Omega \subset \mathfrak{g}^*$ has a natural symplectic structure given as follows (see [CG97, Prop. 1.1.5]):

Assume $f \in \Omega$. The tangent space $T_f\Omega$ at the point f is equal to $\mathfrak{g}/\mathfrak{g}^f$. We define an alternating bilinear form on $\mathfrak{g}$

$$\omega_f \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \quad \omega_f \colon (x,y) \mapsto B_f(x,y) = f([x,y]).$$

The form ω_f descends to $\mathfrak{g}/\mathfrak{g}^f$. Thus the assignment $f \mapsto \omega_f$ gives a non-degenerate 2-form ω on Ω such that $d\omega = 0$.

Notice that $\dim\Omega = \dim(\mathfrak{g}/\mathfrak{g}^f) = \operatorname{rank}(B_f)$ which is an even number. Any coadjoint orbit is an irreducible variety of $\mathfrak{g}^*$ (see [Hum75, Prop. 8.2]).

6 Generalized Weight Modules

In this section, we relax our assumptions and let $\mathfrak{g}$ be an arbitrary finite-dimensional Lie algebra and $\mathfrak{h}$ be a nilpotent Lie subalgebra of $\mathfrak{g}$.

Definition 6.1. *A $\mathfrak{g}$-module N is a generalized weight module over $\mathfrak{h}$, if as an $\mathfrak{h}$-module, N decomposes as*

$$N = \bigoplus_{\mu \in suppN} N^\mu$$

for some subset suppN of $\mathfrak{h}^*$, where N^μ is a non-zero generalized weight space of weight $\mu \in \mathfrak{h}^*$ for each $\mu \in suppN$, i.e. for every $v \in N$ and every $x \in \mathfrak{h}$, there exists $l = l(x,v) \in \mathbb{Z}_{>0}$ such that $(x - \mu(x))^l.v = 0$.

As an $\mathfrak{h}$-module under the adjoint action, $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{h}' \oplus \left(\bigoplus_{\alpha \in \Delta_{\mathfrak{h}} \subset \mathfrak{h}^*} \mathfrak{g}^\alpha \right),$$

where each $\mathfrak{g}^\alpha$ is a generalized weight space of weight $\alpha \in \mathfrak{h}^*$, $\alpha \neq 0$ (i.e., for every $x \in \mathfrak{g}^\alpha$ and every $y \in \mathfrak{h}$ there exists $n \in \mathbb{Z}_{\geq 0}$ so that $(ad(y) - \alpha(y))^n x = 0$), and $\mathfrak{h}'$ is the generalized weight space of weight 0. The set $\Delta_{\mathfrak{h}}$ of all non-zero weights α is the set of $\mathfrak{h}$-roots of $\mathfrak{g}$.

An element $x \in \mathfrak{g}$ is said to act *locally finitely* on a $\mathfrak{g}$-module N if the vector space spanned by the vectors $v, x.v, x^2.v, \ldots$ is finite-dimensional for any $v \in N$. By [Fer90, Cor. 2.7], the set $\mathfrak{g}[N]$ of all elements of $\mathfrak{g}$ which act locally finitely on N is a Lie subalgebra of $\mathfrak{g}$. It is the largest subalgebra of $\mathfrak{g}$ that is locally finite on N and is called the *Fernando subalgebra* of $\mathfrak{g}$ with respect to N. For any $v \in N$, the $\mathfrak{g}[N]$-module $U(\mathfrak{g}[N]).v$ generated by v is finite-dimensional by [PS98, Prop. 1].

Let N be a $\mathfrak{g}$-module, possibly infinite-dimensional. We say that $x \in \mathfrak{g}$ acts *freely* on N if the vectors $v, x.v, x^2.v, \ldots$ are linearly independent for any $v \in N$. If N is a simple $\mathfrak{g}$-module, then any $x \in \mathfrak{g}$ acts either locally finitely or freely on N.

For a nilpotent Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, N is a generalized weight module over $\mathfrak{h}$ precisely when $\mathfrak{h} \subset \mathfrak{g}[N]$. If N is a simple $\mathfrak{g}$-module, then we denote by Γ_N the cone in $\langle \Delta_{\mathfrak{h}} \rangle_{\mathbb{R}}$, the real span of the roots, generated by all $\alpha \in \Delta_{\mathfrak{h}}$ such that $\mathfrak{g}^\alpha$ is not contained in $\mathfrak{g}[N]$.

If $\mathfrak{g}$ is a finite-dimensional solvable Lie algebra and N is a simple $\mathfrak{g}$-module, then $\mathrm{supp} N = \mu + \Gamma_N$, for some $\mu \in \mathfrak{h}^*$ ([PS98, Rem., Prop. 2]). If, in addition, $\mathfrak{g}$ is nilpotent, then the set $\Delta_{\mathfrak{h}}$ is empty by Engel's theorem; therefore $\Gamma_N = 0$. To summarise: if $\mathfrak{g}$ is a nilpotent Lie algebra, $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, and N is a simple $\mathfrak{g}$-module that is a generalized weight module over $\mathfrak{h}$, then $\mathrm{supp} N = \{\mu\}$, for some $\mu \in \mathfrak{h}^*$.

7 Simple Modules Containing an Eigenvector for a Subalgebra

Let $\mathfrak{g}$ be a nilpotent Lie algebra and let $f \in \mathfrak{g}^*$. Assume $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$ subordinate to f, and let N be a $\mathfrak{g}$-module. Thus $f([\mathfrak{k}, \mathfrak{k}]) = 0$. Set

$$\mathfrak{k}(f) = \{x - f(x) \mid x \in \mathfrak{k}\} \subset U(\mathfrak{g}), \tag{1}$$

$$N^{\mathfrak{k},f} = \{n \in N \mid (x - f(x)).n = 0 \ \ \forall x \in \mathfrak{k}\}, \tag{2}$$

and

$$\mathfrak{k}^\top = \{\lambda \in \mathfrak{g}^* \mid \lambda(\mathfrak{k}) = 0\}. \tag{3}$$

The set $f + \mathfrak{k}^\top = \{\lambda \in \mathfrak{g}^* \mid \lambda(x) = f(x) \ \forall x \in \mathfrak{k}\}$ is an affine linear subspace of $\mathfrak{g}^*$. We will show that it is an irreducible algebraic variety. Let $\dim \mathfrak{g} = r$ and $\dim \mathfrak{k} = s$, $s \leq r$. Suppose $e_1, e_2, \ldots, e_r$ is a basis of $\mathfrak{g}$ such that $e_1, e_2, \ldots, e_s$ is a basis of $\mathfrak{k}$. We have the dual basis $e_1^*, e_2^*, \ldots, e_r^*$ of $\mathfrak{g}^*$. Any $\lambda \in \mathfrak{g}^*$ can be uniquely represented by $(a_1, \ldots, a_r) \in \mathbb{C}^r$ where $\lambda = \sum_{1 \leq i \leq r} a_i e_i^*$. Thus

$$\begin{aligned} f + \mathfrak{k}^\top &= \{\lambda \in \mathfrak{g}^* \mid \lambda(e_i) = f(e_i), \ i = 1, \ldots, s\} \\ &= \{(a_1, \ldots, a_s, \ldots, a_r) \mid a_i = f(e_i), \ i = 1, \ldots, s\}. \end{aligned}$$

So the coordinate ring of $f + \mathfrak{k}^\top$ is $\mathbb{C}[a_1, \ldots, a_r]/\langle a_1 - f(e_1), \ldots, a_s - f(e_s)\rangle \simeq \mathbb{C}[\overline{a_{s+1}}, \ldots, \overline{a_r}]$ which does not have zero divisors, hence $f + \mathfrak{k}^\top$ is irreducible (see [CG97, Prop. 2.2.5]). Notice that $\dim(f + \mathfrak{k}^\top) = \dim \mathfrak{k}^\top = \dim(\mathfrak{g}/\mathfrak{k})$.

We will use the following theorem and corollary of Benoist ([Ben90b, Thm. 6.1]). Recall the definitions of lagrangian spaces and varieties from Section 5.

Theorem 7.1. *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra over $\mathbb{C}$, and let $U = U(\mathfrak{g})$. Assume Ω is a G-orbit in $\mathfrak{g}^*$, and I is the primitive ideal of U associated with Ω. Let $f \in \mathfrak{g}^*$ and $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{g}$ such that $f([\mathfrak{k}, \mathfrak{k}]) = 0$. Set $\mathcal{Z} = \Omega \cap (f + \mathfrak{k}^\top)$ and let M be the $\mathfrak{g}$-module $U/(I + U\mathfrak{k}(f))$. Assume S is the set of those simple $\mathfrak{g}$-modules N with annihilator I such that $N^{\mathfrak{k}, f} \neq 0$.*

1. *The following are equivalent:*

   (i) *$\mathcal{Z}$ is a lagrangian subvariety of Ω.*

   (ii) *M is of finite length (has a finite composition series).*

   (iii) *S is a finite set.*

2. *If one (hence all) of the conditions in part 1. hold, then:*

   (a) *$\mathcal{Z}$ is a smooth variety.*

   (b) *There is a bijection between the irreducible components Λ of $\mathcal{Z}$ and the elements M_Λ of S.*

   (c) *There is an isomorphism of $\mathfrak{g}$-modules $M \simeq \bigoplus_\Lambda M_\Lambda^{\oplus m_\Lambda}$, where $m_\Lambda = \dim(M_\Lambda^{\mathfrak{k}, f})$. In particular, m_Λ is finite for each Λ and M is semi-simple.*

Corollary 7.2. *With assumptions as in Theorem 7.1,*

1. *$\mathcal{Z} = \emptyset \Longleftrightarrow M = 0 \Longleftrightarrow S = \emptyset$.*

2. *If $\mathcal{Z}$ is an orbit under the group $K = \exp(\mathfrak{k})$, then M is a multiple of the simple module $M_\mathcal{Z}$, $M = M_\mathcal{Z}^{\oplus m_\mathcal{Z}}$.*

Since $\mathfrak{g}$ is assumed to be a nilpotent Lie algebra, all of its subalgebras are nilpotent too. Let N be a member of the set S as defined in Theorem 7.1. Then the subalgebra $\mathfrak{k}$ acts locally finitely on some element of N, hence on all of N. Thus $\mathfrak{k} \subset \mathfrak{g}[N]$. So N is a generalized weight module over $\mathfrak{k}$ and $\operatorname{supp} N = f|_\mathfrak{k}$.

On the other hand, suppose N is a simple generalized weight module over some subalgebra $\mathfrak{k} \subset \mathfrak{g}$ with $\operatorname{supp} N = \{\mu\}$ relative to $\mathfrak{k}$. Let $I = \operatorname{ann}_{U(\mathfrak{g})}(N)$, and let $\mu = f|_\mathfrak{k}$ for some $f \in \mathfrak{g}^*$. Let $v \in N$ be any non-zero

generalized weight vector of weight μ relative to $\mathfrak{k}$. Then $U(\mathfrak{g}[N])v$ is a finite-dimensional $\mathfrak{g}[N]$-module. Let N_v be a simple $\mathfrak{g}[N]$-submodule of $U(\mathfrak{g}[N])v$. Since $\mathfrak{g}[N] \subseteq \mathfrak{g}$ is nilpotent, N_v is a one-dimensional $\mathfrak{g}[N]$-module, hence a one-dimensional $\mathfrak{k}$-module, and $x \in \mathfrak{k}$ acts by multiplication by $f(x)$ on it, which implies that $f([\mathfrak{k}, \mathfrak{k}]) = 0$ and also that $N^{\mathfrak{k},f} \neq 0$. Thus, we have proved the following result:

Theorem 7.3. *Assume $\mathfrak{g}$ is a finite-dimensional nilpotent Lie algebra over $\mathbb{C}$, and let $U = U(\mathfrak{g})$. Let $I \in Prim\, U, f \in \mathfrak{g}^*$, and $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{g}$ such that $f([\mathfrak{k}, \mathfrak{k}]) = 0$. Let $\mathcal{E}_I^{\mathfrak{k},f}$ be the set of all simple $\mathfrak{g}$-modules N with annihilator I such that $N^{\mathfrak{k},f} \neq 0$, and let $\mathcal{W}_I^{\mathfrak{k},f}$ be the set of all simple $\mathfrak{g}$-modules N with annihilator I which are also generalized weight modules over $\mathfrak{k}$ with $suppN = \{f|_{\mathfrak{k}}\}$. Then*

$$\mathcal{E}_I^{\mathfrak{k},f} = \mathcal{W}_I^{\mathfrak{k},f}.$$

$\square$

As a consequence we have

Corollary 7.4. *With assumptions as in Theorem 7.3, let $\mathfrak{k} = \mathfrak{p}$, a polarisation of f in $\mathfrak{g}$. If Ω is a G-orbit in $\mathfrak{g}^*$ and $f \notin \Omega$, then $\mathcal{E}_I^{\mathfrak{p},f} = \emptyset$, hence $\mathcal{W}_I^{\mathfrak{p},f} = \emptyset$.*

Proof. If Ω is a G-orbit in $\mathfrak{g}^*$ and $f \notin \Omega$, then $f + \mathfrak{p}^\top$ is contained in the coadjoint orbit of f by [Kir99, Prop. 1, Sec. 2]. Since distinct coadjoint orbits are disjoint, $\mathcal{Z} = \Omega \cap (f + \mathfrak{p}^\top) = \emptyset$, which, from Corollary 7.2 and Theorem 7.3 implies that $\mathcal{E}_I^{\mathfrak{p},f} = \mathcal{W}_I^{\mathfrak{p},f} = \emptyset$. $\square$

Next we consider what happens when $\Omega = \Omega_f$, the coadjoint orbit passing through f itself, and $\mathfrak{k} = \mathfrak{p} \in P(f)$, a polarisation of f. Then $\mathcal{Z} = \Omega_f \cap (f + \mathfrak{p}^\top) = f + \mathfrak{p}^\top$, by [Kir99, Prop. 1, Sec. 2]. We want to determine how many irreducible components the variety $\mathcal{Z}$ has when $\mathcal{Z}$ is lagrangian.

8 Induced Modules

Assume $\mathfrak{g}$ is a nilpotent Lie algebra. Let $f \in \mathfrak{g}^*$ be such that $f([\mathfrak{g}, \mathfrak{g}]) \neq 0$. Let $\mathfrak{p} \in P(f)$ be a polarisation of f. Let $\{\mathfrak{p}, f\}$ be the one-dimensional $\mathfrak{p}$-module $\mathbb{C}v$ given by $f|_{\mathfrak{p}}$. By [Dix96, Prop. 6.2.9], the induced $\mathfrak{g}$-module $\mathcal{M}_f := Ind_{\mathfrak{p}}^{\mathfrak{g}}\{\mathfrak{p}, f\} = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \{\mathfrak{p}, f\}$ is simple. The mapping $v \mapsto 1 \otimes v$ of

$\{\mathfrak{p}, f\}$ into $\mathcal{M}_f$ is an injective $\mathfrak{p}$-module homomorphism. So $\{\mathfrak{p}, f\}$ can be identified with a submodule of the $\mathfrak{p}$-module $\mathcal{M}_f$ under this mapping.

As mentioned in Section 2.3, the primitive ideal I associated with the coadjoint orbit passing through f is the precisely the annihilator of the module $\mathcal{M}_f$ in U, and it does not depend on the choice of polarisation of f.

Let $J = \text{ann}_{U(\mathfrak{p})}(\{\mathfrak{p}, f\})$. From [Dix96, Prop. 5.1.7], we have

$$\text{ann}_{U(\mathfrak{g})}(\{\mathfrak{p}, f\}) = U(\mathfrak{g})J,$$

which is a left ideal of $U(\mathfrak{g})$. And, $\text{ann}_{U(\mathfrak{g})}(\mathcal{M}_f) = I$ is the largest two-sided ideal of $U(\mathfrak{g})$ contained in $U(\mathfrak{g})J$.

By Prop. 5.1.9 (i) in [Dix96], the mapping ϕ of $U(\mathfrak{g})$ into $\mathcal{M}_f$ defined by $\phi(u) = u \otimes v$ for all $u \in U(\mathfrak{g})$ is surjective and has kernel $U(\mathfrak{g})J$.

By Prop. 5.1.9 (ii) in [Dix96], the mapping $\bar{\phi}$ of $U(\mathfrak{g})/U(\mathfrak{g})J$ into $\mathcal{M}_f$ inherited from ϕ by passage to the quotient is a $\mathfrak{g}$-module isomorphism. Thus $U(\mathfrak{g})/U(\mathfrak{g})J \simeq \mathcal{M}_f$ as $\mathfrak{g}$-modules, and so by Prop. 5.1.9 (iii) in [Dix96], we see that $U(\mathfrak{g})J = U(\mathfrak{g})\mathfrak{p}(f)$, where

$$\mathfrak{p}(f) = \{x - f(x) \mid x \in \mathfrak{p}\} \subseteq U(\mathfrak{g}).$$

Consequently, $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{p}(f) \simeq \mathcal{M}_f$ and so is a simple $\mathfrak{g}$-module, since $\mathcal{M}_f$ is a simple $\mathfrak{g}$-module. But from Prop. 5.1.7 (ii) in [Dix96], we have $I \subset U(\mathfrak{g})\mathfrak{p}(f)$. Hence $I + U(\mathfrak{g})\mathfrak{p}(f) = U(\mathfrak{g})\mathfrak{p}(f)$ and thus, $U(\mathfrak{g})/(I + U(\mathfrak{g})\mathfrak{p}(f)) = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{p}(f)$ is a simple $\mathfrak{g}$-module.

As shown in Section 7, $\mathcal{Z} = \Omega_f \cap (f + \mathfrak{p}^\top) = f + \mathfrak{p}^\top$ is irreducible, hence it has only one irreducible component. Therefore if $\mathcal{Z}$ is a lagrangian subvariety of Ω_f, then the set $\mathcal{E}_I^{\mathfrak{p},f}$ contains only one element $M_\mathcal{Z}$. We see that $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \{\mathfrak{p}, f\}$ is an element of $\mathcal{E}_I^{\mathfrak{p},f}$, so in this case it is isomorphic to $M_\mathcal{Z}$.

Theorem 8.1. *If $f \in \mathfrak{g}^*$ and the coadjoint orbit Ω_f passing through f is two-dimensional, then for the primitive ideal I associated with Ω_f, the set $\mathcal{E}_I^{\mathfrak{p},f}$ contains only one element, $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \{\mathfrak{p}, f\}$, for any $\mathfrak{p} \in P(f)$.*

Proof. If $\dim \Omega_f = 2$, then $\dim \mathfrak{g} - \dim \mathfrak{p} = 1$, so the irreducible variety $\mathcal{Z} = f + \mathfrak{p}^\top$ is one-dimensional, hence lagrangian. Thus the set $\mathcal{E}_I^{\mathfrak{p},f}$ contains only one element $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \{\mathfrak{p}, f\}$. $\square$

9 Example: $\mathfrak{g} = \mathfrak{n}_3$

Let $\mathfrak{g} = \mathfrak{n}_3$, the Lie algebra of all 3×3 strictly upper triangular matrices over $\mathbb{C}$. We can select a basis $\{x, y, z\}$ for $\mathfrak{g}$, where $x = E_{12}, y = E_{23}, z = E_{13}$ are

standard matrix units. Then, $[x, y] = z$, and $\mathbb{C}z$ is the center of $\mathfrak{g}$, so that $\mathfrak{n}_3$ is isomorphic to the Heisenberg Lie algebra. Since $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] = 0$, for any $f \in \mathfrak{g}^*$ there are only two possibilities:

1. $f([\mathfrak{g}, \mathfrak{g}]) = 0$, or

2. $f([\mathfrak{g}, \mathfrak{g}]) \neq 0$ and $f([[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}]) = 0$.

In Case 1, f defines a one-dimensional $\mathfrak{g}$-module $\mathbb{C}v$ with $x.v = f(x)v$ for all $x \in \mathfrak{g}$.

In Case 2, it is easy to see that any proper subalgebra of $\mathfrak{g}$ is subordinate to f, and that any two-dimensional subalgebra of $\mathfrak{g}$ is a polarisation of f. This means that the coadjoint orbit of f is two-dimensional (because the codimension of the polarisation is 1). Let Ω be a coadjoint orbit in $\mathfrak{g}^*$, and let $I \in \operatorname{Prim} U$ be the corresponding primitive ideal. Now we divide considerations according to whether f belongs to Ω or not.

(i) $f \in \Omega$

Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ and suppose $f([\mathfrak{k}, \mathfrak{k}]) = 0$. Recall that $\mathfrak{k}^\top = \{p \in \mathfrak{g}^* \mid p(\mathfrak{k}) = 0\} \subseteq \mathfrak{g}^*$. Then $\mathcal{Z} = (f + \mathfrak{k}^\top) \cap \Omega \neq \emptyset$, because $f \in \Omega$ and $f \in f + \mathfrak{k}^\top$.

If $\mathfrak{k} = 0$, then $\mathfrak{k}^\top = \mathfrak{g}^*$ and $\mathcal{Z} = \Omega$, so $\mathcal{Z}$ is not lagrangian.

If $\mathfrak{k} = \mathbb{C}z$, then $f + \mathfrak{k}^\top = \Omega$, so $\mathcal{Z} = \Omega$ is not lagrangian.

If $\mathfrak{k} = \mathbb{C}(ax + by + cz)$, where not both a, b are zero, then $\mathcal{Z} = (f + \mathfrak{k}^\perp) \cap \Omega$ is a one-dimensional subvariety of the two-dimensional variety Ω, and so is lagrangian and irreducible. In this case, the set $\mathcal{E}_I^{\mathfrak{k}, f}$ has a unique element.

If $\mathfrak{k}$ is two-dimensional, it is a polarisation of f. Thus $\mathfrak{k} = \mathfrak{p} \in P(f)$, and then $f + \mathfrak{p}^\top \subset \Omega$. Therefore $\mathcal{Z} = f + \mathfrak{p}^\top$ is lagrangian and irreducible, so there is a unique element $M_\mathcal{Z}$ in the set $\mathcal{E}_I^{\mathfrak{p}, f}$ by Theorem 8.1 . Let $\{\mathfrak{p}, f\}$ be the one-dimensional $\mathfrak{p}$-module given by f. Then $M_\mathcal{Z}$ is isomorphic to the induced irreducible $\mathfrak{g}$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \{\mathfrak{p}, f\}$, which we have shown is isomorphic to the simple $\mathfrak{g}$-module $M = U/(I + U\mathfrak{p}(f))$. By part 2 of Corollary 3.2, we have M isomorphic to $M_\mathcal{Z}^{\oplus m_\mathcal{Z}}$. But in this case M is a simple $\mathfrak{g}$-module, so it must be that $m_\mathcal{Z} = 1$.

(ii) $f \notin \Omega$

Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ such that $f([\mathfrak{k}, \mathfrak{k}]) = 0$.

If $\mathfrak{k} = 0$ then $\mathfrak{k}^\top = \mathfrak{g}^*$, and so $\mathcal{Z} = \Omega$, which means $\mathcal{Z}$ is not lagrangian.

If $\mathfrak{k} = \mathbb{C}z$, then $\mathcal{Z} = \emptyset$, so $\mathcal{E}_I^{\mathfrak{k}, f} = \emptyset$.

If $\mathfrak{k} = \mathbb{C}(ax + by + cz)$ with not both a, b zero, then $\mathcal{Z} = (f + \mathfrak{k}^\top) \cap \Omega$ is one-dimensional, hence lagrangian and irreducible. So $\mathcal{E}_I^{\mathfrak{k}, f}$ has a unique element.

If $\mathfrak{k} = \mathfrak{p}$, a polarisation of f, then $(f + \mathfrak{p}^\top) \cap \Omega = \emptyset$, so $\mathcal{Z}$ is empty, hence $\mathcal{E}_I^{\mathfrak{p}, f} = \emptyset$.

10 Lie Superalgebras, Induced Modules and Graded-Primitive Ideals

In this section, we assume that $\mathfrak{g}$ is a finite-dimensional Lie superalgebra over $\mathbb{C}$. Thus $\mathfrak{g}$ has a $\mathbb{Z}_2$-grading, $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, and a bilinear product $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

(1) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{Z}_2$.

(2) $[a, b] = -(-1)^{\alpha\beta}[b, a]$ (graded skew-symmetry)

(3) $[a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]]$, (graded Jacobi identity)

for all $a \in \mathfrak{g}_\alpha, b \in \mathfrak{g}_\beta, c \in \mathfrak{g}_\gamma$. We assume that $\mathfrak{g}$ is nilpotent, so that $\mathfrak{g}^m = [\mathfrak{g}, \mathfrak{g}^{m-1}] = 0$ for some $m \geq 1$. The universal enveloping superalgebra $U(\mathfrak{g})$ is $\mathbb{Z}_2$-graded and is isomorphic to $U(\mathfrak{g}_{\bar{0}}) \otimes \bigwedge(\mathfrak{g}_{\bar{1}})$, where $\bigwedge(\mathfrak{g}_{\bar{1}})$ is the exterior (Grassmann) algebra on $\mathfrak{g}_{\bar{1}}$. We denote $U(\mathfrak{g})$ by $\mathfrak{U}$.

10.1 Graded-Primitive Ideals

A *graded-prime* ideal of $\mathfrak{U}$ is a $\mathbb{Z}_2$-graded ideal P such that for any pair I, J of $\mathbb{Z}_2$-graded ideals of $\mathfrak{U}$ we have $IJ \subseteq P$ only if $I \subseteq P$ or $J \subseteq P$. A *graded-primitive* ideal of $\mathfrak{U}$ is the annihilator of some simple $\mathbb{Z}_2$-graded $\mathfrak{U}$-module. Let GrSpec $\mathfrak{U}$ and GrPrim $\mathfrak{U}$ denote the sets of graded-prime and graded-primitive ideals of $\mathfrak{U}$, respectively, and let Spec U and Prim U denote the sets of prime ideals and primitive ideals respectively of the enveloping algebra U of the Lie algebra $\mathfrak{g}_{\bar{0}}$. Note that GrPrim $\mathfrak{U} \subset$ GrSpec $\mathfrak{U}$ and Prim $U \subset$ Spec U ([Dix96, 3.1.6]).

Below we recount Corollary III of Section 3 in [Let92]:

Proposition 10.1. *Assume that $\mathfrak{g}$ is a finite-dimensional nilpotent Lie superalgebra over $\mathbb{C}$.*

(a) *If P is a graded-prime ideal of $\mathfrak{U}$, then there exists a unique prime ideal $i(P) \in$ Spec U minimal over $U \cap P$.*

(b) *The assignment in part (a) produces homeomorphisms of topological spaces (relative to the Zariski topology):*

$$i : GrSpec \, \mathfrak{U} \to Spec \, U$$

$$i : GrPrim \, \mathfrak{U} \to Prim \, U.$$

The homeomorphism i gives us a bijection between the set of graded-primitive ideals of $\mathfrak{U}$ and the primitive ideals of U.

10.2 Induced modules

Here we relax our assumptions, and let $\mathfrak{g}$ be an arbitrary Lie superalgebra and $\mathfrak{U}$ be its universal enveloping superalgebra. Let $\mathfrak{h}$ a subsuperalgebra of $\mathfrak{g}$, and W a $\mathbb{Z}_2$-graded $\mathfrak{h}$-module, thus a $U(\mathfrak{h})$-module. The induced module $Ind_{\mathfrak{h}}^{\mathfrak{g}} W = \mathfrak{U} \otimes_{U(\mathfrak{h})} W$ inherits a $\mathbb{Z}_2$-grading from W and $\mathfrak{U}$, and has $\mathfrak{U}$-action given by

$$x(u \otimes w) = xu \otimes w, \ x, \ u \in \mathfrak{U}, \ w \in W$$

Now, we impose the assumption that $\mathfrak{g}$ is a nilpotent Lie superalgebra. Define

$$\Lambda = \{\lambda \in \mathfrak{g}^* \mid \lambda(\mathfrak{g}_{\bar{1}}) = 0\} \tag{4}$$

(Then Λ can be identified with $\mathfrak{g}_{\bar{0}}^*$).

Two simple $\mathfrak{g}$-modules (resp. $\mathfrak{g}_{\bar{0}}$-modules) are called *weakly equivalent* if they have the same annihilator in $\mathfrak{U}$ (resp. in U). Let G_0 be the group $\exp(\mathfrak{g}_{\bar{0}})$. By [Dix96, Thm. 6.2.4] there exists a bijective correspondence between the set of G_0-orbits in $\mathfrak{g}_{\bar{0}}^*$ and the set of classes of weakly equivalent $\mathfrak{g}_{\bar{0}}$-modules.

For $\lambda \in \mathfrak{g}^*$, define $\mathfrak{g}^\lambda = \{x \in \mathfrak{g} \mid \lambda([x, y]) = 0, \ \forall y \in \mathfrak{g}\}$. A subsuperalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is said to be *subordinate* to λ if $\lambda([\mathfrak{k}, \mathfrak{k}]) = 0$ and $\mathfrak{g}^\lambda \subset \mathfrak{k}$. A maximal member $\mathfrak{p}$ of the set of subsuperalgebras that are subordinate to λ is called a *polarisation* of λ.

Denote by $\{\mathfrak{k}, \lambda\}$ the one-dimensional $\mathfrak{k}$-module given by λ. Thus, $\{\mathfrak{k}, \lambda\} = \mathbb{C}v$ where $x.v = \lambda(x)v$, for all $x \in \mathfrak{k}$ (i.e. $\{\mathfrak{k}, \lambda\} = V_{\bar{0}} \oplus V_{\bar{1}}$, where $V_{\bar{0}} = \mathbb{C}v$ and $V_{\bar{1}} = 0$). Since $\lambda([\mathfrak{k}, \mathfrak{k}]) = 0$, therefore $\{\mathfrak{k}, \lambda\}$ is a well-defined $\mathfrak{k}$-module. Let $Ind_{\mathfrak{k}}^{\mathfrak{g}} \{\mathfrak{k}, \lambda\}$ denote the induced $\mathfrak{g}$-module $\mathfrak{U} \otimes_{U(\mathfrak{k})} \{\mathfrak{k}, \lambda\}$.

Theorem 10.2. ([Kac77, Sec. 5.2, Thm. $7'(b)$]) *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie superalgebra over an algebraically closed field of characteristic zero. Let $\lambda \in \Lambda$, and let $\mathfrak{p}$ be a polarisation of λ with $\dim \mathfrak{p}_{\bar{0}} = \frac{1}{2}(\dim \mathfrak{g}_{\bar{0}} + \dim(\mathfrak{g}^\lambda)_{\bar{0}})$. Then the following hold:*

- The $\mathfrak{g}$-module $\mathcal{M}_\lambda = Ind_{\mathfrak{p}}^{\mathfrak{g}}\{\mathfrak{p}, \lambda\}$ is simple.

- The map $\lambda \to \mathcal{M}_\lambda$ induces a bijective correspondence between the set of G_0-orbits in Λ and the set of classes of weakly equivalent $\mathbb{Z}_2$-graded simple $\mathfrak{g}$-modules.

11 Results on Graded-Primitive Ideals

Let $\mathfrak{g}$ be a nilpotent Lie superalgebra, $\mathfrak{U}$ (respectively U) be the universal enveloping superalgebra of $\mathfrak{g}$ (universal enveloping algebra of $\mathfrak{g}_{\bar{0}}$). Set $\Lambda = \{\lambda \in \mathfrak{g}^* \mid \lambda(\mathfrak{g}_{\bar{1}}) = 0\} = \mathfrak{g}_{\bar{0}}^*$ as above. An element $\lambda \in \Lambda$ determines the graded-primitive ideal $P_\lambda = \text{ann}_{\mathfrak{U}}(\mathcal{M}_\lambda) \in \text{GrPrim}\,\mathfrak{U}$. Let

$$\Lambda' = \{\lambda \in \Lambda \mid \lambda([\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]) = 0\}. \tag{5}$$

Then we have the following results:

Theorem 11.1. *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a nilpotent Lie superalgebra with universal enveloping superalgebra $\mathfrak{U}$, and let the universal enveloping algebra of $\mathfrak{g}_{\bar{0}}$ be denoted by U.*

(a) *Any $\lambda \in \Lambda'$ gives a graded-primitive ideal P_λ of $\mathfrak{U}$ such that $P_\lambda \cap U$ is a primitive ideal of U. If $\lambda_1, \lambda_2 \in \Lambda'$ are in the same G_0-orbit, then $P_{\lambda_1} = P_{\lambda_2}$.*

(b) *If, in addition, we have $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = 0$, then the map $\lambda \to P_\lambda$ induces a bijection between the set of G_0-orbits in Λ and the set $\text{GrPrim}\,\mathfrak{U}$ of graded-primitive ideals of $\mathfrak{U}$.*

11.1 Proof of Theorem 11.1

Part (a)

Case 1: Suppose $\lambda \in \Lambda'$ is such that $\lambda([\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]) = 0$. Then we have $\lambda([\mathfrak{g}, \mathfrak{g}]) = 0$. So $\mathfrak{g}$ is subordinate to λ. Thus $\mathcal{M}_\lambda = Ind_{\mathfrak{g}}^{\mathfrak{g}}\{\mathfrak{g}, \lambda\} = \{\mathfrak{g}, \lambda\}$. Since $\lambda(\mathfrak{g}_{\bar{1}}) = 0$, the elements of $\mathfrak{g}_{\bar{1}}$ act trivially on $\mathcal{M}_\lambda$, so $\mathcal{M}_\lambda$ is a one-dimensional (therefore simple) $\mathbb{Z}_2$-graded $\mathfrak{g}$-module (the odd subspace of $\mathcal{M}_\lambda$ is trivial). Its annihilator in $\mathfrak{U}$ is the graded-primitive ideal P_λ of $\mathfrak{U}$ generated by all elements of the form $x - \lambda(x)$, $x \in \mathfrak{g}$. Viewed as a member of $\mathfrak{g}_{\bar{0}}^*$, the linear map λ defines a one-dimensional $\mathfrak{g}_{\bar{0}}$-module $N = \mathbb{C}n$ where $x.n = \lambda(x)n$, for all $x \in \mathfrak{g}_{\bar{0}}$. The primitive ideal of U corresponding to this simple $\mathfrak{g}_{\bar{0}}$-module is the two sided ideal Q_λ of U generated by the elements $\{x - \lambda(x) \mid x \in \mathfrak{g}_{\bar{0}}\}$. It is clear that $Q_\lambda = P_\lambda \cap U = i(P_\lambda)$.

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Case 2: Suppose $\lambda \in \Lambda'$ and $\lambda([\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]) \neq 0$. Let $\mathfrak{p} \subset \mathfrak{g}$ be a maximal subalgebra subordinate to λ. Note that since $\lambda([\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]) = 0$ and $\lambda([\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{0}}]) \subset \lambda(\mathfrak{g}_{\bar{1}}) = 0$, we have $\mathfrak{g}_{\bar{1}} \subset \mathfrak{g}^{\lambda} \subset \mathfrak{p}$.

Thus $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where $\mathfrak{p}_{\bar{0}}$ is a maximal subalgebra of $\mathfrak{g}_{\bar{0}}$ subordinate to λ ($\in \mathfrak{g}_{\bar{0}}^*$), which means that $\dim \mathfrak{p}_{\bar{0}} = \frac{1}{2}(\dim \mathfrak{g}_{\bar{0}} + \dim(\mathfrak{g}^{\lambda})_{\bar{0}})$ ([Dix96, 1.12.1]).

Let $\mathcal{M}_{\lambda} = Ind_{\mathfrak{p}}^{\mathfrak{g}}\{\mathfrak{p}, \lambda\}$. This is a simple $\mathfrak{g}$-module. By [Dix96, Thm. 6.1.7, Prop. 6.2.3, Thm. 6.2.4] we have a bijective correspondence

$$Q_{\lambda} \longleftrightarrow G_0\text{-orbit of } \lambda$$

between the primitive ideals of U and the G_0-orbits of elements of $\mathfrak{g}_{\bar{0}}^*$. By Theorem 10.2 we have a bijective correspondence

$$G_0\text{-orbit of } \lambda \longleftrightarrow P_{\lambda}$$

between the set of G_0-orbits in Λ ($= \mathfrak{g}_{\bar{0}}^*$) and the set of classes of weakly equivalent $\mathfrak{g}$-modules. Combining the two, we obtain a correspondence between the primitive ideal Q_{λ} of U associated to the G_0-orbit of λ and the graded-primitive ideal P_{λ} of $\mathfrak{U}$ that is the annihilator in $\mathfrak{U}$ of all the simple $\mathfrak{g}$-modules weakly equivalent to $\mathcal{M}_{\lambda}$. Note that $Q_{\lambda} = \text{ann}_U(\mathcal{N}_{\lambda})$, where $\mathcal{N}_{\lambda} = Ind_{\mathfrak{p}_{\bar{0}}}^{\mathfrak{g}_{\bar{0}}}\{\mathfrak{p}_{\bar{0}}, \lambda\}$, which is a simple $\mathfrak{g}_{\bar{0}}$-module (from [Dix96, Thm. 6.1.1]).

Now, $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, so we may choose linearly independent elements $e_1, e_2, \ldots, e_r \in \mathfrak{g}_{\bar{0}}$ such that $\mathfrak{g}_{\bar{0}} = \langle e_1, e_2, \ldots, e_r \rangle \oplus \mathfrak{p}_{\bar{0}}$ and $\mathfrak{g} = \langle e_1, e_2, \ldots, e_r \rangle \oplus \mathfrak{p}$.

Let $\{\mathfrak{p}, \lambda\} = \mathbb{C}v$ and $\{\mathfrak{p}_{\bar{0}}, \lambda\} = \mathbb{C}w$. Then, the $\mathfrak{g}$-module $\mathcal{M}_{\lambda}$ consists of linear combinations of elements of the type $e_1^{a_1} e_2^{a_2} \cdots e_r^{a_r} \otimes v$ and the $\mathfrak{g}_{\bar{0}}$-module $\mathcal{N}_{\lambda}$ consists of linear combinations of elements of the type $e_1^{a_1} e_2^{a_2} \cdots e_r^{a_r} \otimes w$, where $a_1, a_2, \ldots, a_r$ are non-negative integers.

Since $\mathfrak{g}_{\bar{1}}$ acts trivially on $\mathcal{M}_{\lambda}$, we have that $\mathcal{M}_{\lambda}$ viewed as a $\mathfrak{g}_{\bar{0}}$-module is annihilated by the same elements in U as $\mathcal{N}_{\lambda}$. Therefore $Q_{\lambda} = P_{\lambda} \cap U = i(P_{\lambda})$. Let S_{λ} denote a minimal set of generators of the primitive ideal Q_{λ} of U (we can do this because U is Noetherian), and let T_{λ} denote a minimal set of generators of the graded-primitive ideal P_{λ} of $\mathfrak{U}$. Since $\mathfrak{g}_{\bar{1}}$ acts trivially on $\mathcal{M}_{\lambda}$, we may assume T_{λ} contains the basis elements $\{f_1, \ldots, f_s\}$, of a fixed basis of $\mathfrak{g}_{\bar{1}}$. In fact, we can choose T_{λ} to be such that $T_{\lambda} = S_{\lambda} \cup \{f_1, \ldots, f_s\}$.

If $\lambda_1, \lambda_2 \in \Lambda'$ are in the same G_0-orbit, then $Q_{\lambda_1} = Q_{\lambda_2}$. Then we may suppose S is a minimal set of generators of the primitive ideal $Q_{\lambda_1} = Q_{\lambda_2}$ of U. Then $T_{\lambda_1} = S \cup \{f_1, \ldots, f_s\} = T_{\lambda_2}$ is a minimal set of generators of the graded-primitive ideal P_{λ_i} of $\mathfrak{U}$. Therefore $P_{\lambda_1} = P_{\lambda_2}$.

This completes the proof of Part (a) of Theorem 11.1.

Part (b)

If $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = 0$, then $\Lambda' = \Lambda$. By [Dix96, Sec. 6.1.5, Thm. 6.1.7], each primitive ideal of U is of the form Q_λ, where $\lambda \in \mathfrak{g}_{\bar{0}}^*$. Moreover, if $\mathfrak{k}$ is any maximal subalgebra of $\mathfrak{g}_{\bar{0}}$ subordinate to λ, then $Q_\lambda = \mathrm{ann}_U(Ind_{\mathfrak{k}}^{\mathfrak{g}_{\bar{0}}}\{\mathfrak{k}, \lambda\})$. Thus, the ideal Q_λ depends only on λ.

The map $\lambda \to Q_\lambda$ gives a bijection between the set of G_0-orbits in $\mathfrak{g}_{\bar{0}}^*$ and Prim U ([Dix96, Thm. 6.2.4]).

We see from Cases 1 and 2 above that any $\lambda \in \Lambda' = \Lambda$ determines a graded-primitive ideal P_λ of $\mathfrak{U}$ such that $i(P_\lambda) = P_\lambda \cap U \in$ Prim U. Here, $\Lambda' = \Lambda$, so we can replace Λ' by Λ in the statement above. So, from Proposition 10.1 and Theorem 10.2, the map $\lambda \to P_\lambda$ induces a bijection between the set of graded-primitive ideals of $\mathfrak{U}$ and the set of G_0-orbits in Λ. This completes the proof of Part (b) of Theorem 11.1. $\square$

Before going on, we recall some well-known results:

From [Dix96, Thm. 4.7.9, Sec 4.7.10, and Prop. 6.2.2], we know that for $\lambda \in \mathfrak{g}_{\bar{0}}^*$ that $U/Q_\lambda \simeq \mathcal{A}_n$, the n-th Weyl algebra (Definition 3.1), where $2n$ is the rank of the bilinear form B_λ on $\mathfrak{g}_{\bar{0}}$ defined by $B_\lambda(x, y) = \lambda([x, y])$. Thus there exist elements x_i, y_i, $1 \le i \le n$, in $\mathfrak{g}_{\bar{0}}$, such that $X_i = x_i + Q_\lambda$ $Y_i = y_i + Q_\lambda$ satisfy the Weyl relations in U/Q_λ:

$$[X_i, X_j] = 0 = [Y_i, Y_j]$$

$$[X_i, Y_j] = \lambda([X_i, Y_j]) = \delta_{ij}1.$$

The number n is called the *weight* of the primitive ideal Q_λ. What can we say about the factor $\mathfrak{U}/P_\lambda$? Let us recall the following result from [BM90]:

Theorem 11.2. ([BM90, Cor. B]) *Suppose k is an algebraically closed field of characteristic zero, $\mathfrak{g}$ is a finite-dimensional nilpotent Lie superalgebra over k, and $\mathfrak{U} = U(\mathfrak{g})$ is the universal enveloping superalgebra of $\mathfrak{g}$. If P is a primitive ideal of $\mathfrak{U}$, then*

$$\mathfrak{U}/P \simeq M_s(\mathcal{A}_n(k));$$

and if P is a graded-primitive ideal of $\mathfrak{U}$, then

$$\mathfrak{U}/P \simeq M_s(\mathcal{A}_n(k)) \ or \ \mathfrak{U}/P \simeq M_s(\mathcal{A}_n(k)) \times M_s(\mathcal{A}_n(k)),$$

where $s = 2^m$, m, n are non-negative integers, and $M_s(\mathcal{A}_n(k))$ denotes the algebra of $s \times s$ matrices over the n-th Weyl algebra $\mathcal{A}_n(k)$.

Therefore, as a consequence of Theorem 11.1, we have the following result:

Theorem 11.3. *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie superalgebra over $\mathbb{C}$ with universal enveloping superalgebra $\mathfrak{U}$.*

(a) *Let $\lambda \in \Lambda'$ and suppose P_λ is the corresponding graded-primitive ideal of $\mathfrak{U}$. Then,*

$$\mathfrak{U}/P_\lambda \simeq \mathcal{A}_n$$

where $2n = \text{rank}(B_\lambda)$ on $\mathfrak{g}_{\bar{0}}$, and $\mathcal{A}_n$ is the n-th Weyl algebra over $\mathbb{C}$.

(b) *Suppose $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ satisfies the condition $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = 0$. Then, for any $P \in GrPrim\ \mathfrak{U}$, we have*

$$\mathfrak{U}/P \simeq \mathcal{A}_n$$

for a unique non-negative integer n.

11.2 Proof of Theorem 11.3

(a) Suppose P_λ is the graded primitive ideal corresponding to $\lambda \in \Lambda'$. Then the factor $\mathfrak{U}/P_\lambda$ is constructed by factoring $\mathfrak{U}$ by the relations $T_\lambda = 0$, where T_λ is a minimal set of generators of P_λ. But we can assume that $T_\lambda = S_\lambda \cup \{f_1, \ldots, f_s\}$, where S_λ is a minimal set of generators of the primitive ideal Q_λ of U, and $\{f_1, \ldots, f_s\}$ is a fixed basis of $\mathfrak{g}_{\bar{1}}$. So, taking the relevant Poincaré-Birkhoff-Witt basis of $\mathfrak{U}$, we see that $\mathfrak{U}/P_\lambda \simeq U/Q_\lambda \simeq \mathcal{A}_n$. In the notation of Theorem 11.2, we have $s = 1$ and n is the weight of the primitive ideal of U that is in one-one correspondence with $P \in \text{GrPrim}\ \mathfrak{U}$.

(b) If $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = 0$, then, by Part (b) of Theorem 11.1, any $P \in \text{GrPrim}\ \mathfrak{U}$ is of the form P_λ for some $\lambda \in \Lambda'$. Therefore, by Part (a) above, we have $\mathfrak{U}/P = \mathfrak{U}/P_\lambda \simeq \mathcal{A}_n$, where $2n = \text{rank}\ B_\lambda$ on $\mathfrak{g}_{\bar{0}}$. This proves Part (b). $\square$

12 Applications

12.1 $\mathfrak{g} = \mathfrak{gl}(m, n)^+, m \neq n$

Let $\mathfrak{g} = \mathfrak{gl}(m, n)^+$, where $m \geq 1$, $n \geq 1$, and $m \neq n$. Then $\mathfrak{g}$ is the nilpotent Lie superalgebra of strictly upper triangular matrices in the general linear Lie superalgebra $\mathfrak{gl}(m, n)$ over $\mathbb{C}$. In matrix notation, $\mathfrak{g}$ is defined to be the set of block matrices

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix},$$

where A, D are strictly upper triangular matrices of sizes $m \times m$ and $n \times n$, respectively, and B is an arbitrary $m \times n$ matrix.

The Lie superbracket on $\mathfrak{g}$ is as follows:

$$\left[\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \begin{pmatrix} A' & B' \\ 0 & D' \end{pmatrix} \right] = \begin{pmatrix} AA' - A'A & BD' - B'D + AB' - A'B \\ 0 & DD' - D'D \end{pmatrix}$$

The even part

$$\mathfrak{g}_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mid A \in \mathfrak{n}_m, \ D \in \mathfrak{n}_n \right\}$$

is isomorphic to the nilpotent Lie algebra $\mathfrak{n}_m \times \mathfrak{n}_n$, while the odd part

$$\mathfrak{g}_{\bar{1}} = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid B \in M_{m \times n}(\mathbb{C}) \right\}$$

has dimension mn and $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = 0$. (Note that $\mathfrak{n}_1 = 0$.)

Definition 12.1. *We define an integer-valued function s_i for $i = m$ or n, as follows:*

$$s_i = \begin{cases} \frac{1}{4}(i-2)i & \text{if } i \text{ is even,} \\ \frac{1}{4}(i-1)^2 & \text{if } i \text{ is odd.} \end{cases}$$

Again, by Theorem 11.1, there is a bijection between the set of G_0-orbits in Λ and the set $\mathrm{GrPrim}\ \mathfrak{U}$ of graded-primitive ideals of $\mathfrak{U}$. By Theorem 11.3, for any $P \in \mathrm{GrPrim}\ \mathfrak{U}$, the quotient $\mathfrak{U}/P \simeq A_r$ where r is the weight of the ideal $Q = P \cap U \in \mathrm{Prim}\ U$. In this case, U is the enveloping algebra of the nilpotent Lie algebra $\mathfrak{g}_{\bar{0}} = \mathfrak{n}_m \times \mathfrak{n}_n$. Let U_m and U_n be the universal enveloping algebras of the Lie algebras $\mathfrak{n}_m$ and $\mathfrak{n}_n$, respectively. By Corollary 4.10 in [Muk04], the weights of members of $\mathrm{Prim}\ U_m$ range through $0, 1, \ldots, s_m$, and the weights of members of $\mathrm{Prim}\ U_n$ range through $0, 1, \ldots, s_n$.

For $\lambda \in \Lambda$, any subsuperalgebra $\mathfrak{p}$ of $\mathfrak{g}$ maximally subordinate to λ is of the form $\mathfrak{p}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where $\mathfrak{p}_{\bar{0}}$ is a polarisation in $\mathfrak{g}_{\bar{0}}$ of λ ($\in \mathfrak{g}_{\bar{0}}^*$), as described in the proof of Theorem 11.1. The subalgebra $\mathfrak{p}_{\bar{0}}$ can be chosen to be of the form $\mathfrak{h}_m \times \mathfrak{h}_n$, where $\mathfrak{h}_m$ is a polarisation of $\lambda|_{\mathfrak{n}_m}$ and $\mathfrak{h}_n$ is a polarisation of $\lambda|_{\mathfrak{n}_n}$. So the codimension of $\mathfrak{p}_{\bar{0}}$ in $\mathfrak{g}_{\bar{0}}$ is $r_m + r_n$, where r_m can range through $0, 1, \ldots, s_m$ and r_n can range through $0, 1, \ldots, s_n$ (see remark at the end of Chapter 2 in [Muk04]).

Thus, we have the following:

Proposition 12.2. *If $\mathfrak{U}$ is the universal enveloping superalgebra of $\mathfrak{g} = \mathfrak{gl}(m,n)^+$, then, for any $P \in GrPrim\ \mathfrak{U}$, the quotient $\mathfrak{U}/P \simeq \mathcal{A}_{r_m+r_n}$, where r_m and r_n are unique non-negative integers; $0 \le r_m \le s_m$, and $0 \le r_n \le s_n$, and s_m, s_n are given by Definition 12.1*

12.2 The Heisenberg Lie Superalgebra

The next example, which comes from [BM90, Sec. 0.2(a)], shows that Theorem 11.3 may not hold when $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \neq 0$.

Let $\mathfrak{g}$ be the nilpotent Lie superalgebra over $\mathbb{C}$ with basis for $\mathfrak{g}_{\bar{0}}$ given by x, y, z and basis for $\mathfrak{g}_{\bar{1}}$ given by a, b. Let all Lie superbrackets be zero except $[x, y] = z = -[y, x]$ and $[a, b] = z = [b, a]$. Thus, $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathbb{C}z$. Let $\lambda \in \mathfrak{g}^*$ be such that $\lambda(\mathfrak{g}_{\bar{1}}) = 0$ and $\lambda(z) = 1$. Then $\mathfrak{g}^\lambda = \mathbb{C}z$. The basis elements x, a, z span a subsuperalgebra $\mathfrak{p}$ that is subordinate to λ and is of maximal dimension. Let $\{\mathfrak{p}, \lambda\} = \mathbb{C}v$ denote the one-dimensional $\mathfrak{p}$-module given by λ. By Theorem 10.2, the $\mathbb{Z}_2$-graded $\mathfrak{g}$-module $\mathcal{M}_\lambda = Ind_{\mathfrak{p}}^{\mathfrak{g}}\{\mathfrak{p}, \lambda\}$ is irreducible. We can see that $\mathcal{M}_\lambda$ is spanned by elements of the form

$$y^r b^s \otimes v$$

where r, s are non-negative integers, $r \ge 0$ and $s = 0$ or 1. The annihilator in $\mathfrak{U}$ of this module is the graded-primitive ideal P_λ generated by $z - 1$. But we see that $\mathfrak{U}/P_\lambda \simeq M_2(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{A}_1 \simeq M_2(\mathcal{A}_1)$, because in the quotient $\mathfrak{U}/P_\lambda$, the elements $\bar{x}$ and $\bar{y}$ generate a copy of $\mathcal{A}_1$ and the elements $\bar{1}, \bar{a}, \bar{b}, \bar{a}\bar{b}$ form a basis for the $\mathbb{C}$-algebra $M_2(\mathbb{C})$.

13 Conclusion

In this work we have used results of Benoist, Fernando, Kirillov and Dixmier to study modules and coadjoint orbits of finite-dimensional nilpotent Lie algebras. We have also derived results about certain kinds of simple infinite-dimensional modules and the corresponding graded-primitive ideals of the universal enveloping superalgebra of nilpotent Lie superalgebra, using the work of Bell, Musson, Letzter, and Kac. Our investigations suggest the following problems for future study.

1. Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra, let $f \in \mathfrak{g}^*$, let Ω_f be the coadjoint orbit containing f, and let $\mathfrak{p} \in P(f)$ be a polarisation. If the dimension of Ω_f is greater than two, when is the variety $\Omega_f \cap (f + \mathfrak{p}^\top)$ lagrangian? How many elements does the set $\mathcal{E}_I^{\mathfrak{p},f}$ of simple modules have?

2. Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra over $\mathbb{C}$ with an involution σ (an automorphism of order 2) , and let $\mathfrak{h}$ be the set of fixed points of σ. A simple $\mathfrak{g}$-module is said to be σ-*spherical* if it contains a nontrivial vector annihilated by $\mathfrak{h}$. Let j be the principal anti-automorphism of the universal enveloping algebra U of $\mathfrak{g}$ such that $x \rightarrow -x$ for all $x \in \mathfrak{g}$. Let Prim U be the set of primitive ideals of U. Set $\mathrm{Prim}_\sigma U = \{ I \in \mathrm{Prim}\ U \mid I^\sigma = I^j \}$. In [Ben90a], Benoist showed for σ, a fixed involution on $\mathfrak{g}$, that there is a bijection between the set of isomorphism classes of σ-spherical simple $\mathfrak{g}$-modules and the set of ideals $\mathrm{Prim}_\sigma U$, and he also gave a classification and several constructions of these modules.

Is it possible to develop a theory of a σ-spherical simple modules for finite-dimensional nilpotent Lie *superalgebras* $\mathfrak{g}$ with an involution? Are these modules in one-to-one correspondence with a subset of GrPrim $\mathfrak{U}$, the set of graded-primitive ideals of the universal enveloping superalgebra of $\mathfrak{g}$?

References

[Ben90a] Y. Benoist, *Les modules simples sphériques d'une algèbre de Lie nilpotente (French) [Spherical simple modules of a nilpotent Lie algebra]*, Compositio Math. **73** (1990), 295–327.

[Ben90b] Y. Benoist, *Modules simples sur une algèbre de Lie nilpotente contenant un vecteur propre pour une sous-algèbre (French) [Simple modules over a nilpotent Lie algebra containing an eigenvector for a subalgebra]*, Ann. Sci. École Norm. Sup. (4) **23** (1990), 495–517.

[BM90] A.D. Bell and I.M. Musson, *Primitive factors of enveloping algebras of nilpotent Lie superalgebras*, J. London Math. Soc. (2) **42** (1990), 401–408.

[CG90] L. Corwin and F.P. Greenleaf, *Representations of Nilpotent Lie Groups and Their Applications, Part 1*, Cambridge University Press, New York, NY, 1990.

[CG97] N. Chriss and V. Ginzburg, *Representation Theory and Complex Geometry*, Birkhauser, Boston, 1997.

[Con71] N. Conze, *Sur certaines représentations des algèbres de Lie nilpotentes (French) [On certain representations of nilpotent Lie algebras]*, C. R. Acad. Sci. Paris Sér. A-B **272** (1971), A985–A988.

[Cou95] S. C. Coutinho, *A Primer of Algebraic D-modules*, Cambridge University Press, New York, NY, 1995.

[Dix96] J. Dixmier, *Enveloping Algebras*, Graduate Studies in Mathematics, Amer. Math. Soc., Providence, RI, 1996.

[Fer90] S.L. Fernando, *Lie algebra modules with finite dimensional weight spaces I*, Trans. Amer. Math. Soc. **322** (1990), 757–781.

[FH91] W. Fulton and J. Harris, *Representation Theory*, Springer, New York, 1991.

[Hum72] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer, New York, 1972.

[Hum75] J.E. Humphreys, *Linear Algebraic Groups*, Graduate Texts in Mathematics, Vol. 21, Springer, New York/Berlin, 1975.

[Kac77] V.G. Kac, *Lie superalgebras*, Adv. Math. **26** (1977), 8–96.

[Kac78] V.G. Kac, *Representation of classical Lie superalgebras*, Differential Geometrical Methods in Mathematical Physics II, Lecture Notes in Math., vol. **676**, Springer-Verlag, 1978, pp. 597–626.

[Kir99] A.A. Kirillov, *Merits and demerits of the orbit method*, Bull. Amer. Math. Soc. **36, no.4** (1999), 433–488.

[Kir03] A.A. Kirillov, *Two more variations on the triangular theme*, The Orbit Method in Geometry and Physics (In Honor of A.A. Kirillov) (C. Duval, L. Guieu, and V. Ovsienko, eds.), Birkhauser, 2003.

[Let92] E.S. Letzter, *Prime and primitive ideals in enveloping algebras of solvable Lie superalgebras*, Abelian Groups and Noncommutative Rings, Contemp. Math., vol. **130**, Amer. Math. Soc., Providence, R.I., 1992, pp. 237–255.

[Muk04] S. Mukherjee, *Representations of Nilpotent Lie Algebras and Superalgebras*, Ph.D. Thesis, University of Wisconsin-Madison, 2004.

[Mus92] I.M. Musson, *A classification of primitive ideals in the enveloping algebra of a classical simple Lie superalgebra*, Adv. Math. **91** (1992), 252–268.

[PS98] I. Penkov and V. Serganova, *The support of an irreducible Lie algebra representation*, J. Algebra **209, no.1** (1998), 129–142.

[Sch79] M. Scheunert, *The Theory of Lie Superalgebras: An Introduction*, Springer, Berlin, 1979.